

Q> If n is a natural number and a is integer and a has an inverse in modulo n then $\gcd(a, n) = 1$.

Ans:- Let $d = \gcd(a, n) \neq 1$

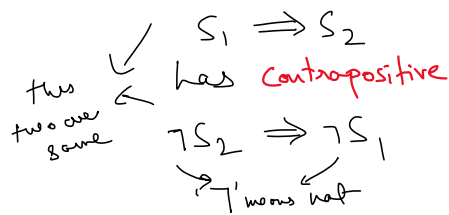
$$d \mid a, d \mid n$$

$$\text{Let } ax = nk + 1, \text{ i.e., } ax \equiv 1 \pmod{n}$$

$$x \equiv a^{-1} \pmod{n}$$

$$d \mid ax, nk$$

$d \mid (ax - nk) = d \mid 1 \Rightarrow$ contradiction. So x doesn't exist.



Q> Let a, b be integers and p be a prime then prove that, $(a+b)^p \equiv a^p + b^p \pmod{p}$

Ans:- $(a+b)^p = a^p + \binom{p}{1} a^{p-1} b + \binom{p}{2} a^{p-2} b^2 + \dots + \binom{p}{p-1} a b^{p-1} + b^p$

$$\equiv a^p + b^p \pmod{p}$$

Q> Let a, m, n be integers and d satisfies, $a^m \equiv 1 \pmod{d}$ and $a^n \equiv 1 \pmod{d}$.

Then show that, $a^{\gcd(m, n)} \equiv 1 \pmod{d}$

Ans:- Using Bezout's lemma we get, $\gcd(m, n) = mx + ny$ for some x, y

Then, $a^{mx+ny} = a^{mx} a^{ny} = (a^m)^x (a^n)^y \equiv 1 \cdot 1 \pmod{d} = 1 \pmod{d}$

Q> Find all integers n such that $|2^n + 5^n - 65|$ is a perfect square

Ans:- $2^{\text{odd}} + 5^{\text{odd}} - 65 =$ even with last digit 2 or 8.

last digit
 $0^2=0, 1^2=1, 2^2=4, 3^2=9, 4^2=6, 5^2=5, 6^2=6, 7^2=9, 8^2=4, 9^2=1$

$$(a_1 a_2 \dots a_n)^2 = b_1 b_2 \dots b_m \Rightarrow a_i^2 \text{'s last digit is } b_m$$

So n must be even

$$2^{2k} = 4^k < 5^k$$

So n must be even

for $n=2k$ and $n \geq 8$ we get,

$$\underbrace{5^{2k}}_{(5^k)^2} < \underbrace{5^{2k} + 2^{2k} - 65}_{\downarrow} < \underbrace{5^{2k} + 2 \times 5^k + 1}_{(5^k + 1)^2}$$

between two consecutive squares, so it is not a perfect square for $n \geq 8$

$$2^{2k} = 4^k < 5^k$$

$n=6 \Rightarrow 5^{2k} - 1$ which is $(5^k)^2 - 1 = (5^k - 1)(5^k + 1)$ so not a perfect square

$n=4 \Rightarrow |6 \cdot 25 + 16 - 65| = 576 = 24^2 \checkmark$

$n=2 \Rightarrow |25 + 4 - 65| = 36 = 6^2 \checkmark$

So $n=2, 4$ are solutions.

Q) If any digit of a 4-digit number is deleted then remaining 3 digit number divides the 4-digit number. How many 4-digit number satisfy this condition?

Ans:- $a_1 a_2 a_3 a_4 = 1000a_1 + 100a_2 + 10a_3 + a_4$

Case 1 $a_1 a_2 a_3 = (100a_1 + 10a_2 + a_3) \mid (1000a_1 + 100a_2 + 10a_3 + a_4)$

$a_1 \neq 0 \Rightarrow (a_1 a_2 a_3) \mid (a_1 a_2 a_3 a_4) - 10(a_1 a_2 a_3)$

$\Rightarrow (a_1 a_2 a_3) \mid a_4 \Rightarrow a_4 = 0$

options = $\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 9 & 10 & 10 & 1 \end{matrix}$

Total = 900

Case 2 $(a_1 a_2 a_4) \mid (a_1 a_2 a_3 a_4) \Rightarrow (a_1 a_2 a_4) \mid (1000a_1 + 100a_2 + 10a_3 + a_4) - (1000a_1 + 100a_2 + 10a_4)$

$\Rightarrow (a_1 a_2 a_4) \mid (10a_3 + 11a_4)$

2 digit \rightarrow so it must be 0

options = $\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 9 & 10 & 1 & 1 \end{matrix}$

Total = 90 subset of case 1

Homework: - Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number. $(a+1) \mid (b-1)$ and $b \mid (a^2 + a + 2)$.

Homework: - Suppose $a, b, d \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $ad = bd \pmod{n}$.
Show that,

$$a \equiv b \pmod{\frac{n}{\gcd(n, d)}}$$