Number Theory 14

24 February 2024 11:35

Q> If n is a natural number and a is integer and a has an inverse in modulo n then $\gcd(a,n)=1$.

Ans:— Let $d=\gcd(a,n) \neq 1$

$d \mid a$, $d \mid n$

$S_1 \Rightarrow S_2$
this two are same
has Contrapositive
$\neg S_2 \Rightarrow \neg S_1$
$\neg$ means not

Let $ax = nk+1$, i.e., $ax \equiv 1 \pmod{n}$
$x \equiv a^{-1} \pmod{n}$

$d \mid ax, nk$
$d \mid (ax-nk) = d \mid 1 \Rightarrow\Leftarrow$ Contradiction. So x doesn't exist.

Q> Let a,b be integers and p be a prime then prove that,
$$(a+b)^p \equiv a^p + b^p \pmod{p}$$

Ans:— $(a+b)^p = a^p + {}^pC_1 a^{p-1}b + {}^pC_2 a^{p-2}b^2 + \cdots + {}^pC_{p-1} a b^{p-1} + b^p$
$\equiv a^p + b^p \pmod{p}$

Q> Let a,m,n be integers and d satisfies,
$a^m \equiv 1 \pmod{d}$ and $a^n \equiv 1 \pmod{d}$.
Then show that, $a^{\gcd(m,n)} \equiv 1 \pmod{d}$

Ans:— Using Bezout's lemma we get, $\gcd(m,n) = mx+ny$ for some x,y

Then,
$$a^{mx+ny} = a^{mx}a^{ny} = (a^m)^x(a^n)^y \equiv 1 \cdot 1 \pmod{n} = 1 \pmod{n}$$

Q> Find all integers n such that $|2^n + 5^n - 65|$ is a perfect square

Ans:— $2^{odd} + 5^{odd} - 65 =$ even with last digit 2 or 8.

last digit
$0^2=0$, $1^2=1$, $2^2=4$, $3^2=9$, $4^2=6$, $5^2=5$, $6^2=6$, $7^2=9$, $8^2=4$, $9^2=1$

$(a_1a_2\ldots a_n)^2 = b_1b_2\ldots b_m \Rightarrow a_n^2$'s last digit is b_m

So n must be even

$2^{2k} = 4^k < 5^k$

So n must be even

for $n=2k$ and $n \geq 8$ we get,

$$2^{2k} = 4^k < 5^k$$

$$\underbrace{5^{2k}}_{(5^k)^2} < \underbrace{5^{2k} + 2^{2k} - 65}_{\Downarrow} < \underbrace{5^{2k} + 2 \times 5^k + 1}_{(5^k+1)^2}$$

between two consecutive squares, so it is not a perfect square for $n \geq 8$

$n=6 \Rightarrow \quad 5^{2k} - 1$ which is $(5^k)^2 - 1 = (5^k-1)(5^k+1)$ so not a perfect square

$n=4 \Rightarrow \quad |625 + 16 - 65| = 576 = 24^2$ ✓

$n=2 \Rightarrow \quad |25 + 4 - 65| = 36 = 6^2$ ✓

So $n = 2, 4$ are solutions.

Q) If any digit of a 4-digit number is deleted then remaining 3 digit number divides the 4-digit number. How many 4-digit number satisfy this condition?

Ans:— $a_1 a_2 a_3 a_4 = 1000a_1 + 100a_2 + 10a_3 + a_4$

Case 1 $\quad a_1 a_2 a_3 = (100a_1 + 10a_2 + a_3) \mid (1000a_1 + 100a_2 + 10a_3 + a_4)$

$a_1 \neq 0$

$\Rightarrow (a_1 a_2 a_3) \mid (a_1 a_2 a_3 a_4) - 10 \times (a_1 a_2 a_3)$

$\Rightarrow (a_1 a_2 a_3) \mid a_4 \quad \Rightarrow a_4 = 0$

$$\begin{array}{lcccc} & a_1 & a_2 & a_3 & a_4 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{options} = & 9 & 10 & 10 & 1 \end{array}$$

Total $= 900$

Case 2 $\quad (a_1 a_2 a_4) \mid (a_1 a_2 a_3 a_4) \Rightarrow (a_1 a_2 a_4) \mid (1000a_1 + 100a_2 + 10a_3 + a_4) - (1000a_1 + 100a_2 + 10a_4)$

$\Rightarrow \underbrace{(a_1 a_2 a_4)}_{\text{3 digit}} \mid \underbrace{(10a_3 + 11a_4)}_{\text{2 digit}}$ → so it must be 0

$$\begin{array}{lcccc} & a_1 & a_2 & a_3 & a_4 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{options} = & 9 & 10 & 1 & 1 \end{array}$$

Total $= 90$ subset of case 1

HomeWork:— Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number. $(a+1) \mid (b-1)$ and $b \mid (a^2+a+2)$.

HomeWork:— Suppose $a, b, d \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $ad \equiv bd \pmod n$

Show that,

$$a \equiv b \left(\bmod \frac{n}{\gcd(n,d)}\right)$$